Table 1 Approach for solving the integral equation of lifting wings—superposition due to Fourier integral

$\ell^*(\xi,\eta,\omega)$;	Solution of Eq. (20):	Flight velocity is unity, out-of-plane motion, simple harmonic in time
$\ell(\xi,\eta,\tau)$;	Through use of Eq. (19):	Flight velocity is unity, w is an arbitrary function in time (or τ)
$\Delta p=\ell\cdot U(t)$;	Through use of Eq. (12):	For variable (nonzero positive) flight velocity

$$\Delta p=\ell\cdot U(t)\equiv\Delta p_1+\Delta p_2+\Delta p_3 \tag{22}$$

The terms on the right hand side of Eq. (22) may be called "quasi-steady term," "instantaneous acceleration term," and "wake term," respectively. These terms are expressed as follows:

$$\frac{\Delta p_1}{\rho U}=\frac{2}{\pi}\int_{-1}^{1}\sqrt{\frac{1-\bar{x}}{1+\bar{x}}}\sqrt{\frac{1+\xi}{1-\xi}}\,\frac{w(\xi,\bar{\tau})}{\bar{x}-\xi}\,\mathrm{d}\xi,\quad \frac{\bar{x}}{\bar{\tau}}\equiv\frac{x}{\tau}\Big/\left(\frac{C_0}{2}\right) \tag{23}$$

$$\frac{\Delta p_2}{\rho U}=\frac{1}{\pi}\int_{-1}^{1}\frac{\mathrm{d}w}{\mathrm{d}\bar{\tau}}(\xi,\bar{\tau})\,\ell n\frac{(\bar{x}-\xi)^2+(\sqrt{1-\bar{x}^2}-\sqrt{1-\xi^2})^2}{(\bar{x}-\xi)^2+(\sqrt{1-\bar{x}^2}+\sqrt{1-\xi^2})^2}\mathrm{d}\xi \tag{24}$$

and

$$\frac{\Delta p_3}{\rho U}=-\frac{2}{\pi}\sqrt{\frac{1-\bar{x}}{1+\bar{x}}}\int_{-1}^{1}\sqrt{\frac{1+\xi}{1-\xi}}\mathrm{d}\xi\int_{\bar{\tau}_s}^{\bar{\tau}}\frac{\mathrm{d}w}{\mathrm{d}\bar{\tau}}(\xi,\bar{\tau})$$
$$\times[\Phi(\bar{\tau}-\bar{\tau}_1)-1(\bar{\tau}-\bar{\tau}_1)]\mathrm{d}\bar{\tau}_1 \tag{25}$$

where $\Phi(\bar{\tau})$ denotes the so-called Wagner function, and $1(\bar{\tau})$ the unit step function. In order to gain physical insight, let us consider a simple case where the airfoil is a flat plate having an incidence α. Then the lift component L_2 resulting from Δp_2 reads

$$L_2/L_{qs}=(n/4)A_g \tag{26}$$

This is positive, when n or acceleration is positive; this is a situation similar to that for a slender delta wing. L_{qs} is the lift component resulting from Δp_1. It is interesting to compare the present result with one obtained by James,[4] which is

$$\ell(s,\tau;A)=\pi\rho\dot{U}(C^2/A^2),\ \text{for}\ t\ll 1,\ \text{and}\ A\gg 1 \tag{27}$$

where α seems to be missing. [See Ref. 4 for symbols in Eq. (27)]. Hence, James' limiting lift is nothing but the present L_2. The last lift component L_3 resulting from Δp_3 is also easily calculated when the Wagner function $\Phi(\tau)$ is expressed by an appropriate approximate form. Since the explicit form of L_3 is somewhat lengthly and space is severely limited, only one figure will be presented. Figure 1 shows an illustrative numerical example, for $n=1$ and chord length $C_0=1$ and 10 m. The unsteady lifts (broken curves) are lower than the quasisteady lifts $L_1=L_{qs}$ (solid curves) except for some short initial periods, where L_2 predominates. It is noteworthy that the unsteadiness corrections are for the most part negative for two-dimensional airfoils, while they are positive for slender wings, for positive accelerations.

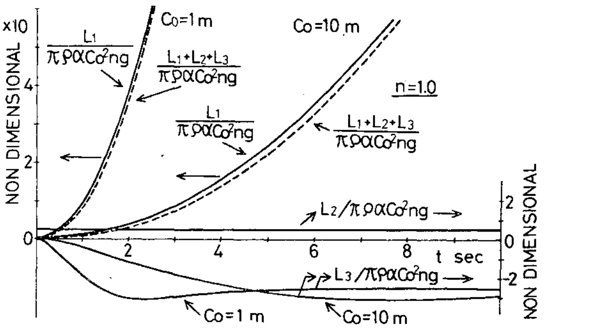

Fig. 1 Lift of flat two-dimensional airfoils flying with constant acceleration (1g) from rest; calculations.

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Explicit Expression for the Smooth Wall Velocity Distribution in a Turbulent Boundary Layer

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Nomenclature

p	= static pressure
u	= mean velocity in x direction
u'	= component of instantaneous fluctuating velocity in x direction
u_0	= wall friction velocity = $\sqrt{\tau_0/\rho}$
u_*	= dimensionless velocity = u/u_0
v	= mean velocity in y direction
v'	= component of instantaneous fluctuating velocity in y direction
w_k	= wake function
x	= Cartesian coordinate in longitudinal direction
y	= Cartesian coordinate perpendicular to wall
y_*	= dimensionless wall distance = yu_0/ν
B	= constant in logarithmic region of mean velocity distribution
C	= constant of proportionality
U_∞	= freestream velocity
δ	= boundary-layer thickness
κ	= von Karman constant
μ	= molecular viscosity
ν	= molecular kinematic viscosity
ν_t	= eddy kinematic viscosity

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ρ = fluid density
τ_0 = wall shear stress
Γ = wake modification function
Π = wake-strength parameter

Superscript

$(\bar{\ })$ = overbar represents time-mean average

Introduction

DEAN[1] recently proposed an expression for the mean velocity distribution over a smooth wall which is valid from the wall to the edge of the boundary layer. The formula combines Spalding's[2] single analytic function for the mean velocity profile in the viscous sublayer, buffer zone, and logarithmic region with Finley's[3] wake function for the outer flow region. The disadvantage of Dean's expression is that it is an implicit function and is therefore difficult to use. Also, the boundary condition at the edge of the layer is not satisfied exactly.

These two problems are overcome in the present work by rejecting Spalding's function in favor of another expression which is not only simple but which also satisfies both the continuity and momentum equations near the wall.

Analysis

Denoting the eddy kinematic viscosity by ν_t, the equation of motion for two-dimensional turbulent flow in the longitudinal direction is:

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\mathrm{d}p}{\mathrm{d}x}+\frac{\partial}{\partial y}\left[(\nu+\nu_t)\frac{\partial u}{\partial y}\right] \tag{1}$$

At the wall, the pressure gradient is balanced by the laminar shear stress gradient,

$$\frac{\mathrm{d}p}{\mathrm{d}x}=\frac{\mu\partial^2 u}{\partial y^2} \tag{2}$$

which leads to

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\frac{\partial}{\partial y}\left(\nu_t\frac{\partial u}{\partial y}\right) \tag{3}$$

for the region very close to the wall. Here, the viscous shear stresses predominate and the velocity profile is linear. The continuity equation therefore implies that $v \sim y^2$ and hence, from Eq. (3),

$$\nu_t \sim y^3 \tag{4}$$

which was first derived by Reichardt.[4]

Use is now made of the experimentally-observed fact that near the wall the longitudinal turbulence intensity, $\sqrt{\overline{u'^2}}$, when nondimensionalized with respect to the wall-friction velocity u_0 varies linearly with the dimensionless wall distance yu_0/ν. This has been verified by Laufer.[5] Hence, as before, $\sqrt{\overline{v'^2}} \sim u_0 y_*^2$, where $y_* = yu_0/\nu$. Since $\nu_t \ \ \partial u/\partial y = \overline{u'v'}$, it follows that

$$\nu_t/\nu \sim y_*^3 \tag{5}$$

as $y_* \to 0$. Away from the wall, $\nu_t/\nu = \kappa y_*$, which results in the familiar derivation of the logarithmic law of the wall:

$$\frac{u}{u_0}=\frac{1}{\kappa}\log_e\frac{yu_0}{\nu}+B \tag{6}$$

The following interpolation formula incorporates these two limiting flow regions in a compact form, as is readily confirmed by inspection:

$$\frac{1}{\nu_t/\nu}=\frac{1}{Cy_*^3}+\frac{1}{\kappa y_*} \tag{7}$$

where C is the constant of proportionality in Eq. (5).

Remembering that in the near-wall region:

$$\frac{\tau_0}{\rho}=(\nu+\nu_t)\frac{\partial u}{\partial y} \tag{8}$$

and putting $u_* = u/u_0$, there results the final expression for the dimensionless velocity gradient which is continuously valid from the wall to the logarithmic region:

$$\frac{\mathrm{d}u_*}{\mathrm{d}y_*}=\frac{\kappa+Cy_*^2}{\kappa+Cy_*^2+C\kappa y_*^3} \tag{9}$$

This will be referred to again shortly.

The outer function has been found by Coles,[6] by analogy with wake flows, to take the following form:

$$\frac{U_\infty-u}{u_0}=\frac{\Pi}{\kappa}(2-w_k)-\frac{1}{\kappa}\log_e\frac{y}{\delta} \tag{10}$$

where w_k is a universal function that has since been endorsed at the Standford Conference,[7] and Π is Coles' profile parameter. An analytical fit to the wake function has been given by Hinze[8] and is often used in the literature:

$$w_k = 1-\cos(\pi y/\delta) \tag{11}$$

A more convenient and equally accurate expression, however, is the polynomial due to Moses[9]:

$$w_k = 6(y/\delta)^2-4(y/\delta)^3 \tag{12}$$

Unfortunately, incorporation of either of these functions into Eq. (10) and subsequent combination with Eq. (6) leads to a discrepancy in the slope condition at the edge of the boundary layer, as discussed by Cornish,[10] Bull,[11] and others. The effect of this discrepancy is most noticeable in cases of strong negative pressure gradient. The problem is overcome by introducing a modification function Γ into Eq. (10):

$$\frac{U_\infty-u}{u_0}=\frac{\Pi}{\kappa}(2-w_k)-\frac{1}{\kappa}\log_e\frac{y}{\delta}-\frac{\Gamma}{\kappa} \tag{13}$$

A polynomial representation for Γ leads to a unique solution that satisfies the outer boundary condition, as first shown by Finley[3] and later by Graville[12]:

$$\Gamma=\left(\frac{y}{\delta}\right)^2\left(1-\frac{y}{\delta}\right) \tag{14}$$

Combining Eqs. (6), (13), and (14) results in the familiar "two-layer" velocity profile but with the new wake function:

$$\frac{u}{u_0}=\frac{1}{\kappa}\log_e\frac{yu_0}{\nu}+B+\frac{\Pi}{\kappa}\left[6\left(\frac{y}{\delta}\right)^2-4\left(\frac{y}{\delta}\right)^3\right]+\frac{1}{\kappa}\left(\frac{y}{\delta}\right)^2\left(1-\frac{y}{\delta}\right) \tag{15}$$

Returning now to Eq. (9), this can be integrated analytically for known values of κ and C. C is found by trial and error by performing a Simpson integration of Eq. (9) in order to match Eq. (6) in the limit as $yu_0/\nu \to \infty$. The value of C is found to be 0.001093 for $\kappa = 0.41$ and $B = 5.0$. The values assigned to the latter two constants are the ones recommended by Coles[7] at

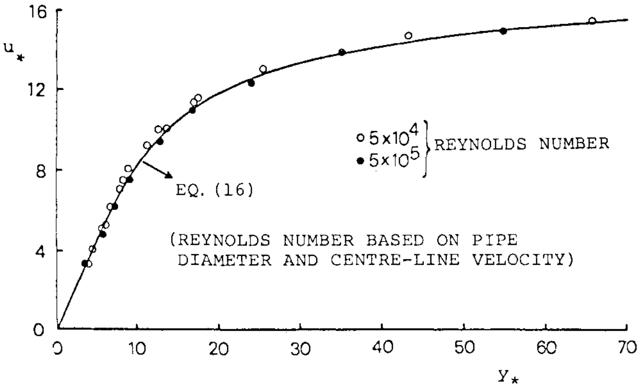

Fig. 1 Comparison of formula with pipe-flow data of Laufer in viscous sublayer and buffer zone.

the Stanford Conference. After integrating Eq. (9) and combining with Eqs. (13) and (14), a good deal of algebra yields the following closed-form expression for the velocity distribution over a smooth wall which is valid continuously from the wall up to the freestream:

$$u_* = 5.424 \tan^{-1}\left[\frac{2y_* - 8.15}{16.7}\right] + \log_{10}\left[\frac{(y_* + 10.6)^{9.6}}{(y_*^2 - 8.15y_* + 86)^2}\right] - 3.52 + 2.44 \times \left\{\Pi\left[6\left(\frac{y}{\delta}\right)^2 - 4\left(\frac{y}{\delta}\right)^3\right] + \left[\left(\frac{y}{\delta}\right)^2\left(1 - \frac{y}{\delta}\right)\right]\right\} \qquad (16)$$

where $u_* = u/u_0$ and $y_* = yu_0/\nu$.

Discussion

As far as the author is aware, there is no other explicit expression available for the smooth-wall velocity distribution which satisfies both the momentum and continuity equations near the wall while satisfying the four boundary conditions: $y=0$, $u=0$ and $du_*/dy_* = 1$; $y=\delta$, $u=U_\infty$ and $du/dy=0$. (For $y=\delta$, $y_* \to \infty$ is regarded as a limiting boundary condition.)

The description of the mean velocity distribution afforded by Eq. (16) is in excellent agreement with Laufer's[5] experimental data near the wall, as is shown in Fig. 1. Away from the wall, as $y \to \infty$, Eq. (16) approaches Eq. (15) asymptotically and so the logarithmic and outer regions are adequately described (see Ref. 7).

Conclusions

It has been shown how the cubic law for the variation in eddy kinematic viscosity very near the wall can be combined with the linear law in the logarithmic region by the use of a simple interpolation formula. This formula leads to an explicit closed-form expression for the velocity distribution over a smooth wall in a turbulent boundary layer which should prove useful in studies of heat and mass transfer and turbulent boundary-layer procedures.

Acknowledgments

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Some Measurements in Radial Free Jets

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Introduction

IN many fluid flow problems of practical importance, one encounters free shear flows such as jets, wakes, and mixing layers. In the class of jet flows there is a flow configuration known as a radial jet which has not yet received much attention. Recently Witze and Dwyer[1] have investigated the turbulent radial jets. In their investigation the radial jets have been classified as "constrained radial jets" and "impinged radial jets." In this investigation a distinct new category of radial jets has been introduced to distinguish it from the other two categories. This is the ideal radial free jet. Essentially it is similar to the impinged jet, but it has small separation distance between the nozzles to avoid the initial developing regions of the axisymmetric jets. The constrained radial jet has been investigated by Heskestad,[2] and some measurements on the ideal radial free jet have been reported by Patel et al.[3]

From these investigations it is noted that the measurements of the impinged radial jets reported so far are limited. For example, Witze and Dwyer selected nozzle spacings greater than 20 times the nozzle diameter, thus limiting the field of investigation to $(r/s) < 0.5$. With such large separation distances the impinged radial jets produced by them were in fact a result of the two interacting axisymmetric turbulent jets that had already undergone some development.

The radial jet investigated by Patel et al.[3] had the constraint ratio of 3.16, and the corresponding separation distance s was $0.316D$. However, their measurements were confined within the region $(r/s) \leq 20$. This limitation was imposed because the fan pressure supplying air to the nozzles was not sufficient to obtain meaningful results beyond

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